Derivations of the Balafoutis & Patel Recursive Dynamics Formulation

Brian Guenter
Charles F. Rose, III

July 2000

Technical Report
MSR-TR-2000-80

Microsoft Research
Microsoft Corporation
One Microsoft Way
Redmond, WA 98052
http://www.research.microsoft.com

In the Siggraph 1996 paper, *Efficient Generation of Motion Transitions Using Spacetime Constraints* [3], we used the Balafoutis and Patel dynamics formulation [1] together with the spacetime optimization animation technique [4] to create high-quality transitions for a full-body human-figure model. Balafoutis and Patel created the most efficient linear-recursive inverse dynamics method currently known. In order to perform gradient-based optimization, such as with BFGS [2], we derived the partials of their equations with respect to joint angle position, velocity, and acceleration.

This technical report details the complete dynamics formulation used in [3], including all the partial equations which could not appear in Siggraph due to space limitations. This tech-report is intended as a companion to that paper.

1 Equations of dynamics & their derivatives

Constants, symbols, and notation:

$\mathbf{o}_i$ = origin of the i-th link coordinate frame.
$\mathbf{c}_i$ = center of mass of the i-th link.
ω_i^i = angular velocity of the i-th link .
$\mathbf{z}_i^i$ = joint axis of the i-thlink expressed in the i-th coordinate frame.
$\mathbf{s}_{i,j}^i$ = vector from $\mathbf{o}_i$ to $\mathbf{o}_j$ expressed in the i-th coordinate frame.
$\mathbf{r}_{i,j}^i$ = vector from $\mathbf{o}_i$ to $\mathbf{c}_j$ expressed in the i-th coordinate frame.
$\mathbf{A}_i$ = 3x3 coordinate (or 4x4 homogeneous) transformation relating the i-th coordinate frame to the $(i-1)$-th frame.
${\mathbf{I}_{c}}_i^k$ = inertia tensor of the i-th link about $\mathbf{c}_i$ expressed in the k-th coordinate frame.

${\mathbf{J}_{c}}_i^k$ = Euler's inertia tensor of the i-th frame about $\mathbf{c}_i$ expressed in the k-th coordinate frame.
$\mathbf{\Omega}_i^i$ = angular acceleration tensor of the i-th link expressed in the i-th coordinate frame.
${\mathbf{F}_{c}}_i^i$ = force vector acting on $\mathbf{c}_i$ expressed in the i-th coordinate frame.
${\mathbf{M}_{c}}_i^i$ = moment vector about $\mathbf{c}_i$ expressed in the i-th coordinate frame.
$\mathbf{f}_i^i$ = force vector exerted on link i by link $(i-1)$.
η_i^i = moment vector exerted on link i by link $(i-1)$.
τ_i = torque at joint i.
$\mathbf{g}$ = gravity.
m_i = mass of the i-th link.

In the above, the subscript indicates the coordinate frame being represented and superscript the coordinate frame in which it is represented.

We use + and - on index variables to denote relative placement in the joint hierarchy. Thus, i_1 is the predecessor of i which is the predecessor of $i+$. For example, in the equation $\omega_{i+}^{i+} = \mathbf{A}_{i+}^T \omega_i^i + \mathbf{z}_{i+}^{i+} \dot{q}_{i+}$, the variable ω_i^i is the angular velocity in the coordinate frame which precedes the coordinate frame of ω_{i+}^{i+}. In other words, coordinate frame i is closer to the root coordinate frame than is frame $i+$. Note that there is no guarantee of a uniquely defined successor.

$$
\begin{aligned}
\mathbf{g} &= [0.0, -9.80655, 0.0]^T \\
\mathbf{J}_{ci}^{\ i} &= \frac{1}{2} trace\left(\mathbf{I}_{ci}^{\ i}\right) 1 - \mathbf{I}_{ci}^{\ i}
\end{aligned}
$$

$$
\begin{aligned}
dual(\mathbf{v}) &= \tilde{\mathbf{v}} = \begin{bmatrix} 0 & -\mathbf{v}_3 & \mathbf{v}_2 \\ \mathbf{v}_3 & 0 & -\mathbf{v}_1 \\ -\mathbf{v}_2 & \mathbf{v}_1 & 0 \end{bmatrix} \\
dual(\tilde{\mathbf{v}}) &= \mathbf{v}
\end{aligned}
$$

Inverse-dynamics equations:

Base conditions at the root of the creature:

$$
\begin{aligned}
\omega_0^0 &= \mathbf{z}_0^0 \dot{q}_0 \\
\dot{\omega}_0^0 &= \mathbf{z}_0^0 \ddot{q}_0 \\
\ddot{\mathbf{s}}_{0,0}^0 &= \mathbf{A}_0^T \mathbf{g}
\end{aligned}
$$

Recursive dynamics equations:

$$
\begin{aligned}
\omega_{i+}^{i+} &= \mathbf{A}_{i+}^T \omega_i^i + \mathbf{z}_{i+}^{i+} \dot{q}_{i+} \\
\dot{\omega}_{i+}^{i+} &= \mathbf{A}_{i+}^T \dot{\omega}_i^i + \tilde{\omega}_i^{i+} \mathbf{z}_{i+}^{i+} \dot{q}_{i+} + \mathbf{z}_{i+}^{i+} \ddot{q}_{i+} \\
\mathbf{\Omega}_{i+}^{i+} &= \dot{\tilde{\omega}}_{i+}^{i+} + \tilde{\omega}_{i+}^{i+} \tilde{\omega}_{i+}^{i+} \\
\ddot{\mathbf{s}}_{0,i+}^{i+} &= \mathbf{A}_{i+}^T \left[\ddot{\mathbf{s}}_{0,i}^i + \mathbf{\Omega}_i^i \mathbf{s}_{i,i+}^i\right] \\
\ddot{\mathbf{r}}_{0,i+}^{i+} &= \mathbf{\Omega}_{i+}^{i+} \mathbf{r}_{i+,i+}^{i+} + \ddot{\mathbf{s}}_{0,i+}^{i+} \\
\mathbf{F}_{ci+}^{\ i+} &= m_{i+} \ddot{\mathbf{r}}_{0,i+}^{i+} \\
\tilde{\mathbf{M}}_{ci+}^{\ i+} &= \left(\mathbf{\Omega}_{i+}^{i+} \mathbf{J}_{ci+}^{\ i+}\right) - \left(\mathbf{\Omega}_{i+}^{i+} \mathbf{J}_{ci+}^{\ i+}\right)^T
\end{aligned}
$$

Backward recursive equations (torque equations):

At a joint controlling an end-effector:

$$\begin{aligned}
\mathbf{f}_i^i &= \mathbf{F}_{c\,i}^{\;i} \\
\eta_I^i &= \tilde{\mathbf{r}}_{i,i}^i \mathbf{F}_{c\,i}^{\;i} + \mathbf{M}_{c\,i}^{\;i} \\
\tau_i &= \eta_i^i \cdot \mathbf{z}_i^i
\end{aligned}$$

At an internal joint:

$$\begin{aligned}
\mathbf{f}_i^i &= \mathbf{F}_{c\,i}^{\;i} + \sum_{i+} \left[\mathbf{A}_{i+} \mathbf{f}_{i+}^{i+} \right] \\
\eta_i^i &= \tilde{\mathbf{r}}_{i,i}^i \mathbf{F}_{c\,i}^{\;i} + \mathbf{M}_{c\,i}^{\;i} + \sum_{i+} \left[\mathbf{A}_{i+} \eta_{i+}^{i+} + \tilde{\mathbf{s}}_{i,i+}^i \mathbf{f}_{i+}^i \right] \\
\tau_i &= \eta_i^i \cdot \mathbf{z}_i^i
\end{aligned}$$

The energy function & the partials:

$$\begin{aligned}
P &= \int_t \sum_i \tau_i^2 dt \\
\frac{\delta P}{\delta \theta_j} &= 2 \int_t \sum_i \frac{\delta \tau_i}{\delta q_j} + \frac{\delta \tau_i}{\delta \dot{q}_j} + \frac{\delta \tau_i}{\delta \ddot{q}_j}
\end{aligned}$$

The recursive partials & their initial conditions:

$$\begin{aligned}
\frac{\delta \omega_{i+}^{i+}}{\delta q_j} &\underset{i+>j}{=} \mathbf{A}_{i+}^T \frac{\delta \omega_i^i}{\delta q_j} \\
\frac{\delta \omega_{i+}^{i+}}{\delta q_j} &\underset{i+=j}{=} \frac{\delta \mathbf{A}_j^T}{\delta q_j} \omega_{j-}^{j-}
\end{aligned}$$

$$\frac{\delta \dot{\omega}_{i+}^{i+}}{\delta q_j} \underset{i+>j}{=} \mathbf{A}_{i+}^T \frac{\delta \dot{\omega}_i^i}{\delta q_j} + \widetilde{\frac{\delta \omega_i^{i+}}{\delta q_j}} \mathbf{z}_{i+}^{i+} \dot{q}_{i+}$$

$$\frac{\delta \dot{\omega}_{i+}^{i+}}{\delta q_j} \underset{i+=j}{=} \frac{\delta \mathbf{A}_j^T}{\delta q_j} \dot{\omega}_{j-}^{j-} + \left(\widetilde{\frac{\delta \mathbf{A}_j^T}{\delta q_j} \dot{\omega}_{j-}^{j-}} \right) \mathbf{z}_j^j \dot{q}_j$$

$$\frac{\delta \mathbf{\Omega}_{i+}^{i+}}{\delta q_j} = \frac{\delta \widetilde{\dot{\omega}_{i+}^{i+}}}{\delta q_j} + \frac{\delta \widetilde{\omega_{i+}^{i+}}}{\delta q_j} \tilde{\omega}_{i+}^{i+} + \left(\frac{\delta \widetilde{\omega_{i+}^{i+}}}{\delta q_j} \tilde{\omega}_{i+}^{i+} \right)^T$$

$$\frac{\delta \omega_{i+}^{i+}}{\delta \dot{q}_j} \underset{i+>j}{=} \mathbf{A}_{i+}^T \frac{\delta \omega_i^i}{\delta \dot{q}_j}$$

$$\frac{\delta \omega_{i+}^{i+}}{\delta \dot{q}_j} \underset{i+=j}{=} \mathbf{z}_j^j$$

$$\frac{\delta \dot{\omega}_{i+}^{i+}}{\delta \dot{q}_j} \underset{i+>j}{=} \mathbf{A}_{i+}^T \frac{\delta \dot{\omega}_i^i}{\delta \dot{q}_j} + \left(\widetilde{\mathbf{A}_{i+}^T \frac{\delta \omega_i^i}{\delta \dot{q}_j}} \right) \mathbf{z}_{i+}^{i+} \dot{q}_{i+}$$

$$\frac{\delta \dot{\omega}_{i+}^{i+}}{\delta \dot{q}_j} \underset{i+=j}{=} \widetilde{\mathbf{A}_j^T \omega_{j-}^{j-}} \mathbf{z}_j^j$$

$$\frac{\delta \mathbf{\Omega}_{i+}^{i+}}{\delta \dot{q}_j} = \frac{\delta \widetilde{\dot{\omega}_{i+}^{i+}}}{\delta \dot{q}_j} + \frac{\delta \widetilde{\omega_{i+}^{i+}}}{\delta \dot{q}_j} \tilde{\omega}_{i+}^{i+} + \left(\frac{\delta \widetilde{\omega_{i+}^{i+}}}{\delta \dot{q}_j} \tilde{\omega}_{i+}^{i+} \right)^T$$

$$\frac{\delta \omega_{i+}^{i+}}{\delta \ddot{q}_j} = 0$$

$$\frac{\delta \dot{\omega}_{i+}^{i+}}{\delta \ddot{q}_j} \underset{i+>j}{=} \mathbf{A}_{i+}^T \frac{\delta \dot{\omega}_i^i}{\delta \ddot{q}_j}$$

$$\frac{\delta \dot{\omega}_{i+}^{i+}}{\delta \ddot{q}_j} \underset{i+=j}{=} \mathbf{z}_j^j$$

$$\frac{\delta \mathbf{\Omega}_{i+}^{i+}}{\delta \ddot{q}_j} = \frac{\delta \widetilde{\dot{\omega}_{i+}^{i+}}}{\delta \ddot{q}_j}$$

$$\frac{\delta \ddot{\mathbf{s}}_{0,i+}^{i+}}{\delta q_j} \underset{i+>j}{=} \mathbf{A}_{i+}^T \left[\frac{\delta \ddot{\mathbf{s}}_{0,i}^{i}}{\delta q_j} + \frac{\delta \mathbf{\Omega}_i^i}{\delta q_j} \mathbf{s}_{i,i+}^i \right]$$

$$\frac{\delta \ddot{\mathbf{s}}_{0,i+}^{i+}}{\delta q_j} \underset{i+=j}{=} \frac{\delta \mathbf{A}_j^T}{\delta q_j} \left[\ddot{\mathbf{s}}_{0,j-}^{j-} + \mathbf{\Omega}_{j-}^{j-} \mathbf{s}_{j-,j}^{j-} \right]$$

$$\frac{\delta \ddot{\mathbf{s}}_{0,i+}^{i+}}{\delta \dot{q}_j} \underset{i+>j}{=} \mathbf{A}_{i+}^T \left[\frac{\delta \ddot{\mathbf{s}}_{0,i}^{i}}{\delta \dot{q}_j} + \frac{\delta \mathbf{\Omega}_i^i}{\delta \dot{q}_j} \mathbf{s}_{i,i+}^i \right]$$

$$\frac{\delta \ddot{\mathbf{s}}_{0,i+}^{i+}}{\delta \dot{q}_j} \underset{i+=j}{=} 0$$

$$\frac{\delta \ddot{\mathbf{s}}_{0,i+}^{i+}}{\delta \ddot{q}_j} \underset{i+>j}{=} \mathbf{A}_{i+}^T \left[\frac{\delta \ddot{\mathbf{s}}_{0,i}^{i}}{\delta \ddot{q}_j} + \frac{\delta \mathbf{\Omega}_i^i}{\delta \ddot{q}_j} \mathbf{s}_{i,i+}^i \right]$$

$$\frac{\delta \ddot{\mathbf{s}}_{0,i+}^{i+}}{\delta \ddot{q}_j} \underset{i+=j}{=} 0$$

$$\frac{\delta \ddot{\mathbf{r}}_{0,i+}^{i+}}{\delta q_j} = \frac{\delta \mathbf{\Omega}_{i+}^{i+}}{\delta q_j} \mathbf{r}_{i,i+}^{i+} + \frac{\delta \ddot{\mathbf{s}}_{0,i+}^{i+}}{\delta q_j}$$

$$\frac{\delta \ddot{\mathbf{r}}_{0,i+}^{i+}}{\delta \dot{q}_j} = \frac{\delta \mathbf{\Omega}_{i+}^{i+}}{\delta \dot{q}_j} \mathbf{r}_{i,i+}^{i+} + \frac{\delta \ddot{\mathbf{s}}_{0,i+}^{i+}}{\delta \dot{q}_j}$$

$$\frac{\delta \ddot{\mathbf{r}}_{0,i+}^{i+}}{\delta \ddot{q}_j} = \frac{\delta \mathbf{\Omega}_{i+}^{i+}}{\delta \ddot{q}_j} \mathbf{r}_{i,i+}^{i+} + \frac{\delta \ddot{\mathbf{s}}_{0,i+}^{i+}}{\delta \ddot{q}_j}$$

$$\frac{\delta \mathbf{M}_{c\,i+}^{\;i+}}{\delta q_j} = \frac{\delta \mathbf{\Omega}_{i+}^{i+}}{\delta q_j} \mathbf{J}_{c\,i+}^{\;i+} - \left(\frac{\delta \mathbf{\Omega}_{i+}^{i+}}{\delta q_j} \mathbf{J}_{c\,i+}^{\;i+} \right)^T$$

$$
\begin{aligned}
\frac{\delta \mathbf{M}_{c}{}_{i+}^{i+}}{\delta \dot{q}_j} &= \frac{\delta \mathbf{\Omega}_{i+}^{i+}}{\delta \dot{q}_j}\mathbf{J}_{c}{}_{i+}^{i+} - \left(\frac{\delta \mathbf{\Omega}_{i+}^{i+}}{\delta \dot{q}_j}\mathbf{J}_{c}{}_{i+}^{i+}\right)^T \\
\frac{\delta \mathbf{M}_{c}{}_{i+}^{i+}}{\delta \ddot{q}_j} &= \frac{\delta \mathbf{\Omega}_{i+}^{i+}}{\delta \ddot{q}_j}\mathbf{J}_{c}{}_{i+}^{i+} - \left(\frac{\delta \mathbf{\Omega}_{i+}^{i+}}{\delta \ddot{q}_j}\mathbf{J}_{c}{}_{i+}^{i+}\right)^T
\end{aligned}
$$

$$
\frac{\delta \mathbf{F}_{c}{}_{i+}^{i+}}{\delta q_j} = m_{i+}\frac{\delta \ddot{\mathbf{r}}_{0,i+}^{i+}}{\delta q_j}
$$

$$
\frac{\delta \mathbf{F}_{c}{}_{i+}^{i+}}{\delta \dot{q}_j} = m_{i+}\frac{\delta \ddot{\mathbf{r}}_{0,i+}^{i+}}{\delta \dot{q}_j}
$$

$$
\frac{\delta \mathbf{F}_{c}{}_{i+}^{i+}}{\delta \ddot{q}_j} = m_{i+}\frac{\delta \ddot{\mathbf{r}}_{0,i+}^{i+}}{\delta \ddot{q}_j}
$$

The backward-recursive partials:

$$
\begin{aligned}
\frac{\delta \mathbf{f}_i^i}{\delta q_j} &\underset{\exists i+}{=} \frac{\delta \mathbf{F}_{c}{}_i^i}{\delta q_j} + \sum_{i+}\left[\frac{\delta \mathbf{A}_{i+}}{\delta q_j}\mathbf{f}_{i+}^{i+} + \mathbf{A}_{i+}\frac{\delta \mathbf{f}_i^i}{\delta q_j}\right] \\
\frac{\delta \mathbf{f}_i^i}{\delta q_j} &\underset{\nexists i+}{=} \frac{\delta \mathbf{F}_{c}{}_i^i}{\delta q_j}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta \mathbf{f}_i^i}{\delta \dot{q}_j} &\underset{\exists i+}{=} \frac{\delta \mathbf{F}_{c}{}_i^i}{\delta \dot{q}_j} + \sum_{i+}\left[\mathbf{A}_{i+}\frac{\delta \mathbf{f}_i^i}{\delta \dot{q}_j}\right] \\
\frac{\delta \mathbf{f}_i^i}{\delta \dot{q}_j} &\underset{\nexists i+}{=} \frac{\delta \mathbf{F}_{c}{}_i^i}{\delta \dot{q}_j}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta \mathbf{f}_i^i}{\delta \ddot{q}_j} &\underset{\exists i+}{=} \frac{\delta \mathbf{F}_{c}{}_i^i}{\delta \ddot{q}_j} + \sum_{i+}\left[\mathbf{A}_{i+}\frac{\delta \mathbf{f}_i^i}{\delta \ddot{q}_j}\right] \\
\frac{\delta \mathbf{f}_i^i}{\delta \ddot{q}_j} &\underset{\nexists i+}{=} \frac{\delta \mathbf{F}_{c}{}_i^i}{\delta \ddot{q}_j}
\end{aligned}
$$

$$\frac{\delta \eta_i^i}{\delta q_j} \underset{\exists i+}{=} \widetilde{\mathbf{r}_{i,i+}^i} \frac{\delta \mathbf{F}_{ci}^{\,i}}{\delta q_j} + \frac{\delta \widetilde{\mathbf{M}_{ci}^{\,i}}}{\delta q_j} + \sum_{i+} \left[\frac{\delta \mathbf{A}_{i+}}{\delta q_j} \eta_{i+}^{i+} + \mathbf{A}_{i+} \frac{\delta \eta_{i+}^{i+}}{\delta q_j} + \widetilde{\mathbf{s}_{i,i+}^i} \frac{\delta \mathbf{A}_{i+}}{\delta q_j} \mathbf{f}_{i+}^{i+} + \widetilde{\mathbf{s}_{i,i+}^i} \mathbf{A}_{i+} \frac{\delta \mathbf{f}_{i+}^{i+}}{\delta q_j} \right]$$

$$\frac{\delta \eta_i^i}{\delta q_j} \underset{\nexists i+}{=} \widetilde{\mathbf{r}_{i,i+}^i} \frac{\delta \mathbf{F}_{ci}^{\,i}}{\delta q_j} + \frac{\delta \widetilde{\mathbf{M}_{ci}^{\,i}}}{\delta q_j}$$

$$\frac{\delta \eta_i^i}{\delta \dot{q}_j} \underset{\exists i+}{=} \widetilde{\mathbf{r}_{i,i+}^i} \frac{\delta \mathbf{F}_{ci}^{\,i}}{\delta \dot{q}_j} + \frac{\delta \widetilde{\mathbf{M}_{ci}^{\,i}}}{\delta \dot{q}_j} + \sum_{i+} \left[\mathbf{A}_{i+} \frac{\delta \eta_{i+}^{i+}}{\delta \dot{q}_j} + \widetilde{\mathbf{s}_{i,i+}^i} \mathbf{A}_{i+} \frac{\delta \mathbf{f}_{i+}^{i+}}{\delta \dot{q}_j} \right]$$

$$\frac{\delta \eta_i^i}{\delta \dot{q}_j} \underset{\nexists i+}{=} \widetilde{\mathbf{r}_{i,i+}^i} \frac{\delta \mathbf{F}_{ci}^{\,i}}{\delta \dot{q}_j} + \frac{\delta \widetilde{\mathbf{M}_{ci}^{\,i}}}{\delta \dot{q}_j}$$

$$\frac{\delta \eta_i^i}{\delta \ddot{q}_j} \underset{\exists i+}{=} \widetilde{\mathbf{r}_{i,i+}^i} \frac{\delta \mathbf{F}_{ci}^{\,i}}{\delta \ddot{q}_j} + \frac{\delta \widetilde{\mathbf{M}_{ci}^{\,i}}}{\delta \ddot{q}_j} + \sum_{i+} \left[\mathbf{A}_{i+} \frac{\delta \eta_{i+}^{i+}}{\delta \ddot{q}_j} + \widetilde{\mathbf{s}_{i,i+}^i} \mathbf{A}_{i+} \frac{\delta \mathbf{f}_{i+}^{i+}}{\delta \ddot{q}_j} \right]$$

$$\frac{\delta \eta_i^i}{\delta \ddot{q}_j} \underset{\nexists i+}{=} \widetilde{\mathbf{r}_{i,i+}^i} \frac{\delta \mathbf{F}_{ci}^{\,i}}{\delta \ddot{q}_j} + \frac{\delta \widetilde{\mathbf{M}_{ci}^{\,i}}}{\delta \ddot{q}_j}$$

$$\frac{\delta \tau_i}{\delta q_j} = \frac{\delta \eta_i^i}{\delta q_j} \cdot \mathbf{z}_i^i$$

$$\frac{\delta \tau_i}{\delta \dot{q}_j} = \frac{\delta \eta_i^i}{\delta \dot{q}_j} \cdot \mathbf{z}_i^i$$

$$\frac{\delta \tau_i}{\delta \ddot{q}_j} = \frac{\delta \eta_i^i}{\delta \ddot{q}_j} \cdot \mathbf{z}_i^i$$

References

[1] Balafoutis, C. A., and Patel, R. V. *Dynamic Analysis of Robot Manipulators: A Cartesian Tensor Approach.* Kluwer Academic Publishers, 1991.

[2] Gill, P. E., Murray, W., and Wright, M. H. *Practical Optimization.* Academic Press, 1981.

[3] Rose, C. F., Guenter, B., Bodenheimer, B., and Cohen, M. Efficient generation of motion transitions using spacetime constraints. In *Computer Graphics* (Aug. 1996), pp. 147–154. Proceedings of SIGGRAPH 96.

[4] Witkin, A., and Kass, M. Spacetime constraints. In *Computer Graphics* (Aug. 1988), pp. 159–168. Proceedings of SIGGRAPH 88.